Math 255A' Lecture 25 Notes

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1 Weakly Compact Operators

1.1 Weak compactness and reflexivity

In this lecture, X, Y, etc. will be real[1] Banach spaces. We will write B_X as the closed unit ball in $\|\cdot\|_X$.

Definition 1.1. $T \in \mathcal{B}(X,Y)$ is **weakly compact** if $\overline{T(B_X)}^{\mathrm{wk}(Y)}$ is is weakly compact in Y.

We will start with a bit of a digression. Suppose we have a Banach space X. We can embed it inside its dual X^** by $x \mapsto \widehat{x}$. The weak topology of X is the restriction of the weak* topology on X^{**} to X. We will denote by τ the weak* topology on X^{**}.

Proposition 1.1. *Let X be a Banach space, and let τ be the weak* topology on X^{**}. Then $\overline{B_X}^{\tau} = B_{X^{**}}$; i.e. B_X is τ-dense in $B_{X^{**}}$.*

Proof. Let $C := \overline{B_X}^{\tau} \subseteq B_{X^{**}}$. Suppose that $z \in B_{X^{**}} \setminus C$. Then, by Hahn-Banach, there exists a continuous linear functional f on (X^{**}, τ) and $\alpha \in \mathbb{R}$ such that $f(C) \leq \alpha < \alpha+\varepsilon \leq f(z)$. That is, there is a continuous linear functional on X such that

$$C(f) \leq \alpha < \alpha + \varepsilon \leq z(f)$$

Moreover, $C(f)$ contains a neighborhood of 0. By rescaling f, we can take $\alpha = 1$. Then $C(f) := \{y(f) : y \in C\} \supseteq \{f(x) : x \in B_X\}$. What this says is that $\|f\|_{X^*} \leq 1$. However, since $z(f)$ is the pairing of elements in the unit balls of their respective spaces, we should not have $z(f) > 1$, □

Corollary 1.1. *X is dense in X^{**}.*

Corollary 1.2. *X is reflexive if and only if B_X is weakly compact.*

[1]The story is not so different for the complex case.

Proof. ($\Longrightarrow$): This is Banach-Alaoglu.

($\Longleftarrow$): If B_X is weakly compact, then $B_X \subseteq X^{**}$ is compact for τ. So B_X is closed in (X^{**}, τ). Then $B_X = \overline{B_X}^{\tau} = B_{X^{**}}$. $\square$

We can rephrase this corollary as the following:

Corollary 1.3. *X is reflexive if and only if I_X is weakly compact.*

Proposition 1.2. *If X or Y is reflexive, then every $T \in \mathcal{B}(X, Y)$ is weakly comapct.*

Proof. Consider $\overline{T(B_X)}^{Y} \subseteq Y$; we want to show that this is weakly compacts. If X is reflexive, then B_X is comapct, so $T(B_X)$ is weakly compact. On the other hand, if Y is reflelxvibe, $r(B_Y)$ is compact for all r. Now take r large enough so that $\overline{T(B_X)}^{Y} \subseteq rB_Y$. $\square$

Proposition 1.3. *If S or T is weakly compact, so is $S \circ T$.*

This is the same proof as before.

1.2 Characterization of weak compactness

Corollary 1.4. *$T \in \mathcal{B}(X, Y)$ is weakly compact if it has a factorization*

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "R"'] & Y \\ W \arrow[ur, "S"'] & \end{tikzcd}$$

where W is reflexive.

Theorem 1.1. *This is an exact characterization of weak compactness.*

Proof. Every T has the factorization

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "Q"'] & Y \\ X/\ker T \arrow[ur, "\overline{T}"'] & \end{tikzcd}$$

where $T(B_X) = \overline{T}(B_{X/\ker T})$. So it is enough to treat $\overline{T}$. So we may assume that $\ker T = \{0\}$.

Switch to regarding $X \subseteq Y$ with different norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, where $\|\cdot\|_Y|_X \lesssim \|\cdot\|_X$ (meaning there is an implicit constant in the inequality). We will find a W and $\|\cdot\|_W$ with $X \leq W \leq Y$ such that $(W, \|\cdot\|_W)$ is reflexive, $\|\cdot\|_Y|_W \lesssim \|\cdot\|_W$, and $\|\cdot\|_W|_X \leq \|\cdot\|_X$.

The idea here comes from the theory of **interpolated Banach spaces**. For $w \in Y$, let $p_n(w) := \inf\{2^{-n}\|x\|_X + 2^n\|y\|_W : x \in X, y \in Y, x + y = w\}$.[2] These are new norms on Y. Let

$$p(w) := \sqrt{\sum_n p_n(w)^2}, \qquad W := \{w : p(w) < \infty\}.$$

Check that

1. The p_n satisfy the triangle inequality, so p does, too. Then p is a norm on W, and (W, p) is a normed space. Moreover, W is a Banach space.
2. If $x \in X$, then $p_n(x) \leq 2^{-n}\|x\|_X$, so $p(x) \lesssim \|x\|_X$.
3. If $w \in W$, then $p_1(w) \leq p(w)$. So there exists a decompostion $w = x + y$ such that $\|x\|_X + \|y\|_W \leq p(w)$. So $\|w\|_Y = \|x + y\|_Y \lesssim p(w)$.

To finish, we will show that W is reflexive. What is the dual of (W, p)? We claim that $f \in W^*$ if and only if there is a sequence $(f_n)_n \in Y^*$ such that $f(w) = \sum_n f_n(w)$ for all w and $\sum_n p_n^*(f_n)^2 < \infty$, where p_n^* is the dual norm on Y^* induced by p_n.

Let $Y_n = (Y, p_n)$. Then W is isometrically isomorphic to a subspace $\{(y_n)_n \in \bigoplus_{L^2} Y_n : y_n = y_m \; \forall n, m\}$. Check that the dual of $\bigoplus_{L^2} Y_n$ is $\bigoplus_{L^2} Y_n^*$. So W^* is the quotient $(\bigoplus_{L^2} Y_n^*)/W^\perp$. This proves the claim.

To show that W is reflexive, we will show that I_W is weakly compact. Now suppose $(z_j)_j \subseteq B_W$; we want to find a weakly convergent subsequence in W. We may assume that $p(z_j) < 1$ for all j. Write $z_j = x_{j,n} + y_{j,n}$ such that $2^{-n}\|x_{j,n}\|_X + 2^n\|y_{j,n}\|_Y$ is very close to $p_n(z_j)$. In particular, $p_n(z_j) < 1$ for every n, so $\|x_{j,n}\|_X \leq 2^n$ and $\|y_{j,n}\|_Y \leq 2^{-n}$. Because B_X is weakly precompact in Y, for each n, there is a $y_n \in Y$ such that $x_{j,n} \xrightarrow{\text{wk}} y_n$ as $j \to \infty$. We also have that for each j, $\|x_{j,n} - x_{j,m}\|_Y = \|y_{j,n} - y_{j,m}\| \leq 2^{-n} + 2^{-m}$. Taking $j \to \infty$, we get $\|y_n - y_m\|_y \leq 2^{-n} + 2^{-m}$ (weak limits cannot increase norm). So $y_n \xrightarrow{\|\cdot\|_Y} y \in Y$ as $n \to \infty$. So we get that $z_j \to y$ (weakly in Y); check this from the definition.

To finish, we need $y \in W$, and we need to show that $z_j \to y$ weakly in W. The point is that for each n, $p_n(y) \leq \liminf_j p_n(z_j)$. Then Fatou's lemma gives $p(y) \leq \liminf_j p(z_j) < \infty$. So $y \in W$. Now for $f(y) = \sum_n f_n(y) = \sum_n \lim_j f_n(z_j)$. We can take out the limit outside the sum because $|f_n(y)| \leq p_n^*(f_n)p_n(y)$., which is a uniform bound. $\square$

[2] Imagine you can pay for $x \in X$ with $\|\cdot\|_X$ and $y \in Y\|$ with $\|\cdot\|_Y$. Then this is the least you have to pay for w.